Polish Spaces

— Chirantan Mukherjee

— (X, τ) top space
τ open sets.
(X, τ), <u>G_δ set</u> if they are countable intersection of open sets

F_σ - set if they are countable union of closed sets

<u>Basis</u> $\mathcal{B} \subset X$
- $x \in X$, $\exists B \in \mathcal{B}$ st $x \in B$
- $x \in B_1 \cap B_2$, $\exists B_3 \in \mathcal{B}$ st $x \in B_3 \subseteq B_1 \cap B_2$

<u>Product topology</u>
$(X_i, \tau_i)_{i \in I}$ family of top spaces.
$\prod_{i \in I} X_i$ is the topology whose open sets looks like $\prod_{i \in I} U_i$ where $U_i \in X_i$ except for finitely many i's $\in I$ $U_i = X_i$

$$\mathcal{B} = \{ \prod B_i \mid B_i \in X_i \}$$

— X as <u>metrizable</u> if $\exists$ a metric d st τ is a topology of (X, d)

d is compatible with a τ

d is compatible with τ

<u>Note</u> if d is a compatible metric with τ

$$\Rightarrow d' = \frac{d}{1+d} \text{ is also compatible}$$

—. $(X_n, d_n)_n$

$\left(\prod_{n \in \mathbb{N}} X_n, d \right)$

$$\rightarrow d(x,y) = \sum_{n=0}^{\infty} 2^{-(n+1)} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

where $x = (x_n)_n$
$y = (y_n)_n$

—. A top space X to be <u>second countable</u> if there is a countable basis for it's topo

—. If X is a top space, we say X is <u>T_1</u> if every singleton of X is closed.

if X is a top space, we say X is <u>regular</u> if for any $x \in X$ and an open nbd N of x, $\exists$ open nbd U of x st $\overline{U} \subseteq N$

—. $D \subseteq X$ is <u>dense</u> if it meets
every nonempty open sets

every non-empty open sets.

— X is <u>separable</u> if it has a countable dense subset.

Thm :- A second countable space is separable.

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(X,d), there $\exists$ $(\hat{X},\hat{d})$ which is complete st (X,d) is a subspace of $(\hat{X},\hat{d})$ and X is dense in $\hat{X}$. This space $(\hat{X},\hat{d})$ is unique upto isometry and we call it is a <u>completion</u> of (X,d)

$$(X,d_X) \xrightarrow{f} (Y,d_Y)$$
$$d_Y(f(x),f(y)) = d_X(x,y)$$

Thm $\hat{X}$ is separable iff X is separable

<u>Defn</u> :- A top space (X,τ) is called <u>completely metrizable</u> if it admits a compatible metric d st (X,d) is complete

Defn (Polish Space):-

A top space (X,τ) is called a <u>Polish space</u> if it is (1) <u>separable</u> and

② Completely metrizable

<u>Note</u>:- separable metric space
↳ completion is Polish

→ Ex $\mathbb{R}, \mathbb{C}$

Thm| The class of
→ ① Completely metrizable space is closed under countable products and sums

② Polish spaces are closed under countable products and countable sums

Proof// $(X_i)_{i\in I}$ is completely metrizable.

① $\prod X_i$ is metrizable if I is countable. We can assume X_i's are completely metrizable,

if $((x_i^n)_{i\in I})_{n\in\mathbb{N}}$ is Cauchy

$\Rightarrow (x_i^n)_{n\in\mathbb{N}}$ is Cauchy $\forall\, i\in I$.

$\Rightarrow (x_i^n)_{n\in\mathbb{N}} \to x_i \in X_i$

$\Rightarrow ((x_i^n)_{i\in I})_{n\in\mathbb{N}} \to (x_i)_{i\in I}$

$\Rightarrow \prod X_i$ is completely metrizable.

② <u>Fact</u>:- the class of separable spaces is closed under countable products
" " " sum.

$\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$

$(0,1)$, $[0,1]$, $\mathbb{S}^1$

$$\underbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \dots \mathbb{S}^1}_{n \text{ times}} = \mathbb{T}^n \quad \text{torus}$$

$\hookrightarrow$ Polish

$\mathbb{S}^1 \times \mathbb{S}^1 \times \dots$ infinite dimensional torus

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A is discrete top space, completely metrizable & if A is also countable it is Polish

$\Rightarrow A^{\mathbb{N}}$ is Polish

$\{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$ Cantor space is Polish

$\mathbb{N}^{\mathbb{N}} = \mathcal{N}$ Baire space

Kuratowski Thm

Thm: If X is a metrizable space.

$A \underset{\text{closed}}{\subseteq} X \Rightarrow A$ is G_δ set

Proof Let d be the metric on X

$$A = \bigcap_{n \in \mathbb{N}} \{x \in X \mid d(x, A) < \tfrac{1}{n+1}\}$$

$\Rightarrow \mathring{A}$ is G_δ set $\because x \to d(x, A)$ is continuous

(X, d) metric space & $A \subseteq X$ diameter

$\text{diam}(A) := \sup\{d(x,y) \mid x, y \in A\}$

(X, τ) be a top space. (Y, d) be a ms and $f : A \to Y$, $A \subseteq X$

$\text{osc}_f(x) := \inf\{\text{diam}(f(U \cap A)) \mid U \text{ is open nbd of } x \in X\}$

Note:- If $x \in \mathring{A} \Rightarrow x$ is a continuity point of $f \Leftrightarrow \text{osc}_f(x) = 0$

Kuratowski Theorem:- If X is a metrizable space, Y is a completely metrizable space. $A \subseteq X$

$f : A \to Y$ C^0

$\Rightarrow$ we can find a G_δ subset $\tilde{G}$ of X st $A \subseteq G \subseteq \bar{A}$ and $g : G \to Y$ is a C^0 extension of f.

Proof Note

Thm:- (If X is metrizable and $Y \subseteq X$ be completely metrizable $\Rightarrow Y$ is a G_δ set in X.)

Conversely, if X is a completely metrizable space and $Y \subseteq X$ is G_δ set $\Rightarrow Y$ is completely metrizable.

Proof// follows from Kuratowski's Theorem

Consider the $Id_Y : Y \to Y$

This is $\mathcal{C}^0$ so $\exists$ is a G_δ set G

$Y \subseteq G \subseteq \overline{Y}$, and $g : G \to Y$ is a $\mathcal{C}^0$ extension of Id_Y. $\because Y$ is dense in G $\Rightarrow g = Id_G$

$\Rightarrow Y = G$

Conversely,

let $Y = \bigcap_{n \in \mathbb{N}} U_n$ where U_n are open sets in X. $F_n = X \setminus U_n$

d be a complete compatible metric on X

d' on X st $d' = d(x,y)$

$$\sum_{n=0}^{\infty} \min \left\{ 2^{-(n+1)}, \left| \frac{1}{1-d(x,F_n)} - \frac{1}{1-d(y,F_n)} \right| \right\}$$

This metric is compatible with the topo on Y.

To Show: (Y,d') is complete

(y_i) be Cauchy in (Y,d')

$\Rightarrow$ it is also Cauchy in (X,d)

$\Rightarrow y_i \to y$ in X

But, $\forall n$ $\lim_{i,j\to\infty} \left| \frac{1}{1-d(y_j,F_n)} - \frac{1}{1-d(y_i,F_n)} \right| = 0$

So, $\forall n$, $\frac{1}{d(y_i,F_n)}$ converges in $\mathbb{R}$

$\Rightarrow d(y_i,F_n)$ is diverges from 0

$\because d(y_i,F_n) \longrightarrow d(y,F_n)$

we have $d(y,F_n) \neq 0 \quad \forall n$

$\Rightarrow y \notin F_n \quad \forall n$

$\Rightarrow y \in Y$

$y_i \longrightarrow y$ in (Y,d')

Baire Category

Defn :- (X,τ) be top space. $A \subseteq X$, nowhere dense if $\text{Int}(\overline{A}) = \phi$

→ Cantor set is nowhere dense in $\mathbb{R}$

$\mathbb{Z}$ is nowhere dense in $\mathbb{R}$

$\mathbb{R}$ " " " " $\mathbb{R}^2$

Defn | (X,τ) be a top space. $A \subseteq X$ is meager (1st category) if it is a countable union of nowhere dense set

$$A = \bigcup_{n\in\mathbb{N}} \{ A_n \mid \text{Int}(\overline{A_n}) = \phi \}$$

non-meager / 2^{nd} category.

ex $\{a\}$ is always non-meager

$\mathbb{C}$ is meager in $\mathbb{R}$

$\mathbb{Q}$ is meager in $\mathbb{R}$

$\mathbb{Q} = \{q_1, q_2, \dots\} = \bigcup_{i=1}^{\infty} \{q_i\}$

Defn/ (X, τ), A is meager

A^c to be comeager (or residual)

ex $\mathbb{Z}$ is residual in $\mathbb{Z}$

Baire Space

A top space (X, τ) is called a Baire space if

(a) every non-empty open set in X is non-meager

b) every residual set in X is dense

c) intersection of countably many dense open sets in X is dense.

$\mathbb{R}$
$\mathbb{C}$ is Baire

$\mathbb{Q}$ is Not a Baire space

Baire Category Thm :- Every completely metrizable space is Baire.

Every locally compact Hausdorff space is Baire.

Proof// Let X be a completely metrizable space and let d be a complete metric on X compatible with the topology on X.

A_n dense open sets of X.

$\bigcap_{n \in \mathbb{N}} A_n$ is dense, for any non-empty open subset A of X

$$\bigcap_{n \in \mathbb{N}} (A \cap A_n) \neq \phi$$

$$A \cap \bigcap_{n \in \mathbb{N}} A_n$$

$\because$ A is non-empty open set, it contains a ball B_1 with radius < 1 st $\overline{B_1} \subset A$

$\because$ A_1 is dense and B_1 is open, $B_1 \cap A_1 \neq \phi$ and is open $\because$ both B_1 and A_1 are open.

As $B_1 \cap A_1$ is non-empty set, it contains a ball B_2 of radius $< \frac{1}{2}$ st $\overline{B_2} \subset B_1 \cap A_1$

$\forall n > 1$ and B_n is an open ball of radius $< \frac{1}{n}$ with $\overline{B_n} \subset B_{n-1} \cap A_{n-1}$

$\because$ A_n is dense and B_n is open, $B_n \cap A_n \neq \phi$

and it contains open ball B_{n+1} of radius $\frac{1}{n+1}$ with $B_{n+1} \subset B_n \subset A_n$.

$\Rightarrow B_{n+1} \subset B_n \quad \forall n \in \mathbb{N}$

Let, x_i to be the center of B_i, we have $d(x_j, x_i) < \frac{1}{i} \quad \forall j > i$

and hence x_i is a Cauchy seq.

$\because$ (X,d) is a complete metric space then is $x \in X$ st $x_i \to x$. For any m, $\exists$ i_0 st $i \geq i_0$ $d(x_i, x) < \frac{1}{m}$

Hence, $x \in B_m = \bigcap_{n=1}^{m} B_n$

$\therefore x \in \bigcap_{n\in\mathbb{N}} B_n \subset A \cap \bigcap_{n\in\mathbb{N}} A_n$

$\Rightarrow \bigcap_{n\in\mathbb{N}} A_n$ is dense

$\Rightarrow X$ is Baire space.

— X be a locally compact H

Note

Baire Measurability

Defn : Let X be a set. Then a σ-ideal on X is a collection $\mathcal{I}$ of subsets of X st :-

(i) if $A \subset B$ and $B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$

a) if $A \subseteq$ ...

b) if $A_1, A_2, \dots \in \mathcal{I} \Rightarrow \exists\, B$ st $B \in \mathcal{I}$
and $\bigcup A_i \subseteq B$

c) $\phi \in \mathcal{I}$

Let $\mathcal{I}$ be a σ-ideal on a set X. If $A, B \subseteq X$, we say A, B are equal modulo $\mathcal{I}$ $A =_{\mathcal{I}} B$ if the

$$A \Delta B := (A \setminus B) \cup (B \setminus A) \in \mathcal{I}$$

If $\mathcal{I}$ is a σ-ideal of meager sets $A =^* B$ we say A, B are equal modulo meager sets

Defn (Baire Property BP) :- (X, τ) top space. $A \subseteq X$, A has the BP if $A =^* U$ for some $U \underset{\text{open}}{\subseteq} X$

Theorem :- X be a top space. The class of sets having BP is a σ-algebra on X.
Infact it is the smallest σ-algebra containing all open and meager sets.

Proof// If U is open $\Rightarrow \overline{U} \setminus U$ is closed nowhere dense and so is meager.

Moreover
If F is closed $\Rightarrow F \setminus (\text{Int } F)$ is closed nowhere dense.

$\Rightarrow U =^* \overline{U}$ and $F =^* \text{Int}(F)$

If A has the BP, so that $A =^* U$ for some open $U \Rightarrow \underline{X \setminus A} =^* X \setminus U =^* \text{Int}(X \setminus U)$

$\Rightarrow X \setminus A$ has the BP.

Finally if each A_n has the BP, $A_n =^* U_n$ where U_n is open $\Rightarrow \bigcup_n A_n =^* \bigcup_n U_n$

$\Rightarrow \bigcup_n A_n$ has the BP.

$A =^* U$ where U is open

$\Rightarrow M = A \triangle U$, M is meager

$A = M \triangle U$

<u>Note</u>:- All (Borel set) F_σ and G_δ set have the BP. But the converse is NOT true.

(There are set with BP that are not Borel)

Thm:- X be a top space, $A \subseteq X$. Then the following are equiv:-

a) A has the BP

b) $A = G \cup M$ where G is G_δ
M is meager

c) $A = F \setminus M$ where F is F_σ
M is meager

Proof Note

Baire Measurable Fn

Defn/ X, Y be top spaces and $f: X \to Y$, we say f is Baire measurable if the inverse image of any open subset of Y has the BP in X.

Note:- Y is second countable, it is clear enough to only consider the inverse images of a countable basis of Y.

Thm:- X, Y be top spaces and $f: X \to Y$ be Baire measurable. If Y is second countable, $\exists$ a set $G \subseteq X$, i.e. the intersection of dense open sets st $f|_G$ is $\mathcal{C}^0$.
In particular if X is Baire, f is is $\mathcal{C}^0$ on a dense G_δ set.

Proof// Let $\{U_n\}$ be a basis of Y.

$\Rightarrow f^{-1}(U_n)$ has a BP on X, so let V_n be open in X and let F_n be countable union of closed nowhere dense sets with $f^{-1}(U_n) \Delta V_n \subseteq F_n$.

$\Rightarrow G_n = X \setminus F_n$ is a countable intersection of dense open sets

$$G = \bigcap_n G_n$$

$\because f^{-1}(U_n) \cap G = V_n \cap G$

$\Rightarrow f|_G$ is $\mathcal{C}^0$

Kuratowski - Ulam Theorem

Thm 1 / Let X be a top space. Y be a second countable space. $S \subseteq X \times Y$, $x \in X$ and $S_x := \{ y \in Y \mid (x,y) \in S \}$

vertical section of S at x

(a) If S is nowhere dense $\Rightarrow S_x$ is nowhere dense in Y for comeagerly many $x \in X$

b) If S is meager $\Rightarrow S_x$ is meager in Y for comeagerly many $x \in X$

Proof // (a) $Y \neq \phi$ and S is closed.

Let U be the complement of S.

It is enough to show that U_x is dense for comeagerly many $x \in X$.
Let $\{Y_n\}$ be a basis for the topology of Y made of non-empty sets.
$U_n := \text{proj}_X(U \cap (X \times Y_n))$ is dense open in X.
$x \in \bigcap_{n \in \mathbb{N}} U_n \Rightarrow U_x \cap Y_n$ is not empty $\forall n$

$\Rightarrow U_x$ is dense

(b) Same.

<u>Thm 2</u> / Let X, Y be countable space. $A \subseteq X$ and $B \subseteq Y$. $\Rightarrow A \times B$ is meager $\Leftrightarrow$ A or B is meager

<u>Proof</u> // If $A \times B$ is meager and A is not meager $\Rightarrow \exists x \in X$ st $(A \times B)_x = B$ is meager by Thm 1

Conversely, if A is meager and $A = \bigcup_{n \in \mathbb{N}} N_n$ where N_n is nowhere dense
$\Rightarrow A \times B = \bigcup_{n \in \mathbb{N}} N_n \times B$, so it is enough to show that $N_n \times B$ is nowhere

enough to show ... dense. This follows from the fact that if U is dense open in X $\Rightarrow$ $U \times Y$ is dense open in $X \times Y$.

(Kuratowski - Ulam Thm): X, Y be second countable spaces and $S \subseteq X \times Y$ having the BP

a) S_x has the BP for comeagerly many $x \in X$.

$S^y := \{x \in X \mid (x,y) \in S\}$ has the BP for comeagerly many $y \in Y$.

b) S is meager is equivalent to S_x is meager for comeagerly many $x \in X$ and S^y is meager for comeagerly many $y \in Y$

c) S is comeager is equivalent to S_x is comeager for comeagerly many $x \in X$ and to S^y is comeager for comeagerly many $y \in Y$.

Proof// U be an open set and M be a meager set with $S \Delta U \subseteq M$

a) $\forall x \in X$, $S_x \Delta U_x \subseteq M_x$

By Thm 1 S_x has the BP for

comeagerly many $x \in X$.

(b) By Thm 1, if S is meager $\Rightarrow$
S_x is meager for comeagerly many $x \in X$.
Conversely, if S is not meager $\Rightarrow U$ is not meager, which gives open sets $V \subseteq X$ and $W \subseteq Y$ st $V \times W \subseteq U$

$V \times W$ is not meager.

By Thm 2 V, W are not meager
$\Rightarrow x \in V$ st S_x and M_x are meager, as $W \setminus M_x \subseteq U_x \setminus M_x$

$\subseteq S_x$

$W \subseteq S_x \cup M_x$

is meager $\Rightarrow \Leftarrow$

(c) from (b).